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Note

Regular relations and bicartesian squares

R. Banach
Computer Science Department, University of Manchester, Manchester, M13 9PL, UK

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Abstract

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It is shown that regular relations, which arise in a number of areas of programming theory, can be characterised in a variety of ways as pullbacks in $\mathcal{S}et$; and up to isomorphism, as bicartesian squares in $\mathcal{S}et$.

1. Introduction

The purpose of this short note is to advertise the universal properties of regular relations. A regular relation turns out to be a pullback in $\mathcal{S}et$ (in many different ways), and a bicartesian square in $\mathcal{S}et$ (uniquely up to isomorphism).

Regular relations arise more or less explicitly in a number of areas. For instance, the works of Mili and coworkers, have highlighted their usefulness in various aspects of program development, see e.g. [5, 6] and references therein. They also arise in the study of specifications, and in the theory of data reification [3]. Similar relations also arise when one asks the questions: "under what conditions is $A \rightarrow P \leftarrow B$ a pushout of something" or, "under what conditions is $A \leftarrow K \rightarrow B$ a pullback of something", as in

Correspondence to: R. Banach, Computer Science Department, University of Manchester, Manchester, M13 9PL, UK. Email: banach@cs.man.ac.uk.

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[2]. It seems inevitable that there are going to be many corners of computer science where similar phenomena occur and where the usefulness of regular relations is going to be rediscovered independently in one form or another. An obvious candidate for this might well be the theory of relational databases for instance. By bringing out the connection with category theory, we hope to both promulgate further the usefulness of categorical modes of thinking in computer science, and also to short-circuit some of the ad-hoc methods of reasoning about regular relations that have characterised their use so far.

In the next section we review some known properties of regular relations and present some additional criteria for regularity. Section 3 shows the connection with pullbacks and bicartesian squares in $\mathcal{S}et$. Section 4 concludes this paper.

2. Regular relations

A relation R from a set A to a set B is a subset of $A \times B$. Many useful and well-known properties of relations can be found in [5–9].

We write $a \cdot R$ for the image of a under R, and $\hat{R}$ for the transpose or inverse of R. We write $R * S$ for the relational composition of R and S.

The kernel of R, written $K(R)$, is given by

$$a\, K(R)\, a' \Leftrightarrow \emptyset \neq \mathbb{A}' \cdot R \subseteq a \cdot R.$$

The nucleus of R, written $N(R)$, is given by

$$N(R) = R * \hat{R}.$$

A relation R is regular iff

$$R * \hat{R} * R = R,$$

which reduces to $R * \hat{R} * R \subseteq R$ because $R \subseteq R * \hat{R} * R$ anyway.

A relation R is uniform iff for all $a, a' \in A$

$$a \cdot R \cap a' \cdot R \neq \emptyset \;\Rightarrow\; a \cdot R = a' \cdot R.$$

A relation R is rational, or difunctional, iff there are partial functions $f: A \to P$, $g: B \to P$ such that

$$R = f * \hat{g}.$$

A relation R is 3-closed iff for all $A' \subseteq A$, $B' \subseteq B$ with $|A'| \leqslant 2$, $|B'| \leqslant 2$,

$$|\{(a,b) \mid a \in A',\ b \in B',\ (a,b) \in R\}| \neq 3.$$

The proofs of the following results are straightforward.

Theorem 2.1. *Let R be a relation from A to B. Then the following conditions are equivalent.*

(1) *R is regular.*
(2) *R is uniform.*
(3) *R is rational.*
(4) *R is 3-closed.*
(5) $K(R)=N(R)$.

The above is a mild extension of results in Mili [5], Jaoua [6], who also restrict to the case $A=B$. It offers a wide variety of ways of looking at regular relations.

Corollary 2.2. *Let R be regular. Then $\hat{R}$ is regular.*

Corollary 2.3. *Let R be a regular relation from A to B. Let $f: A' \to A$ and $g': B' \to B$ be (partial) functions. Then $R' = f' * R * \hat{g}'$ is regular.*

Theorem 2.4. *Let R be a relation from A to B. R is regular iff there is a bijection $\sim$ between equipolent partitions of* $\mathrm{dom}(R)$ *and* $\mathrm{cod}(R)$ *such that*

$$[a] \sim [b] \Leftrightarrow a\,R\,b,$$

where $[a]$ is the block of the partition of $\mathrm{dom}(R)$ *containing a, and $[b]$ is the block of the partition of* $\mathrm{cod}(R)$ *containing b.*

3. Regular relations as pullbacks and bicartesian squares

Section 2, which showed that there is a large number of ways of characterising regular relations, gives strong indication that they have some deep properties. Generally, the clearest way of presenting such deep properties is via category theory, which exalts structure and relationship at the expense of concrete detail, as far as this is possible. Accordingly we recast the properties of regular relations in categorical terms. We work in $\mathcal{S}et$, the category of sets and (total) functions between sets.

A bicartesian square in a category is a commuting square which is both a pushout and a pullback. Bicartesian squares are also called Dolittle diagrams according to Barr and Wells [4] and pulation squares according to Adamek et al. [1]. Their universal properties are illustrated in Fig. 1.

$\mathcal{S}et$ has pullbacks of all pairs $f: A \to P$, $g: B \to P$. Up to isomorphism the formulae below give appropriate $s: K \to A$, $t: K \to B$.

$$K = \{(a,b) \mid \exists\, p \in P \text{ such that } f(a) = p = g(b)\}, \qquad (*)$$

$$s((a,b)) = a, \qquad t((a,b)) = b.$$

Likewise $\mathcal{S}et$ has pushouts of all pairs $s: K \to A$, $t: K \to B$. Up to isomorphism, the following construction gives suitable $f: A \to P$, $g: B \to P$.

$$
\begin{tikzcd}
 & & & P' \\
 & A \arrow[r, "f"] \arrow[urr, "f'"] & P \arrow[ur, dotted, "\theta"] & \\
 & K \arrow[u, "s"] \arrow[r, "t"] & B \arrow[u, "g"] \arrow[uur, "g'"'] & \\
K' \arrow[uur, "s'"] \arrow[ur, dotted, "\psi"] \arrow[urr, "t'"'] & & &
\end{tikzcd}
$$

Fig. 1.

Let $\approx_K$ be the smallest equivalence relation on K such that $k_1 \approx_K k_2$ if $s(k_1)=s(k_2)$ or $t(k_1)=t(k_2)$. Let $[k]_K$ be the equivalence class containing k and $[K]_K$ be the set of equivalence classes of $\approx_K$.

$$P=(A-s(K))\uplus[K]_K\uplus(B-t(K)), \qquad (**)$$

$$f:A\to P:\ a\mapsto\begin{cases}[k]_K & \text{if } a=s(k),\\ a & \text{otherwise,}\end{cases}$$

$$g:B\to P:\ b\mapsto\begin{cases}[k]_K & \text{if } b=t(k),\\ b & \text{otherwise.}\end{cases}$$

Theorem 3.1. *If $s:K\to A$, $t:K\to B$ is a pullback of $f:A\to P$ and $g:B\to P$, then we can construct a regular relation R from A to B such that K is isomorphic to R as relations from A to B. Conversely if R is a regular relation from A to B we can construct a pullback square $s:K\to A$. $t:K\to B$, $f:A\to P$, $g:B\to P$ such that K and R are isomorphic.*

Proof. If we have functions $f:A\to P$ and $g:B\to P$ then $R=f*\hat{g}$ is regular by rationality: and from the explicit formula $(*)$ for the pullback object K given above, we see that $K=R$. Consequently, since pullbacks are unique up to isomorphism, any other pullback object will be isomorphic to R as required.

Conversely, let R be a regular relation. Then by rationality, R can be written as $R=\bar{f}*\hat{\bar{g}}$ for some partial functions $\bar{f}:A\to\bar{P}$, $\bar{g}:B\to\bar{P}$. However, to construct a pullback we need total functions. There are many ways of getting total functions with the required behaviour, but we choose the following, for reasons to be discussed below.

$$C=\operatorname{cod}(\bar{f})\cap\operatorname{cod}(\bar{g}),$$

$$P=(A-\bar{f}^{-1}(C))\uplus C\uplus(B-\bar{g}^{-1}(C)), \qquad (***)$$

$$f: A \to P: a \mapsto \begin{cases} \bar{f}(a) & \text{if } a \in \bar{f}^{-1}(C) \\ a & \text{otherwise,} \end{cases}$$

$$g: B \to P: b \mapsto \begin{cases} \bar{g}(b) & \text{if } b \in \bar{g}^{-1}(C) \\ b & \text{otherwise.} \end{cases}$$

By construction, $\text{cod}(f) \cap \text{cod}(g) = \text{cod}(\bar{f}) \cap \text{cod}(\bar{g})$ whence $R = \bar{f} * \hat{\bar{g}} = f * \hat{g}$. Since f and g are total functions, they have a pullback $s: K \to A$, $t: K \to B$; so if K is given by the explicit formula $(*)$ above, we easily see that $K = R$, and any other pullback of f and g is isomorphic to R as required. □

As we noticed above, there are many acceptable sets P other than the one given in $(***)$, and function pairs $f: A \to P$, $g: B \to P$, such that the pullback of f and g is $s: K \to A$, $t: K \to B$, since the explicit pullback object $K = R$ depends only on the restriction of $\bar{f}$ to $\bar{f}^{-1}(\text{cod}(\bar{g}))$ and on the restriction of $\bar{g}$ to $\bar{g}^{-1}(\text{cod}(\bar{f}))$. We can use this freedom to stipulate that $f: A \to P$ and $g: B \to P$ is actually the pushout of $s: K \to A$ and $t: K \to B$. But this is exactly what $(***)$ gives us, as a comparison of $(**)$ and $(***)$ elucidates. We rapidly find:

Theorem 3.2. If $s: K \to A$, $t: K \to B$, $f: A \to P$, $g: B \to P$ is a bicartesian square, then we can construct a regular relation R from A to B. Conversely if R is a regular relation from A to B we can construct a bicartesian square, $s: K \to A$, $t: K \to B$, $f: A \to P$, $g: B \to P$, unique up to isomorphism, such that K and R are isomorphic as relations from A to B.

4. Conclusions

The regularity condition $R = R * \hat{R} * R$ is well-known in algebra (if $*$ and $\hat{}$ are suitably interpreted), and there is a wealth of material on such topics as regular rings, and regular operator algebras. Our concern in this note was with regular relations, and we have demonstrated that they correspond to unique bicartesian squares in $\mathcal{S}et$ up to isomorphism. References have been given in the introduction which show that regular relations are very useful in many areas of computer science. Their categorical properties should help to emphasise their potential for useful application, and should streamline the technical aspects of their use in such applications.

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